Dynamics of fluctuating magnetic fields in turbulent dynamos incorporating ambipolar drifts

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ABSTRACT

Turbulence with a large magnetic Reyonolds number, generically leads to rapidly growing magnetic noise over and above any mean field. We revisit the dynamics of this fluctuating field, in homogeneous, isotropic, helical turbulence. Assuming the turbulence to be Markovian, we first rederive, in a fairly transparent manner, the equation for the mean field, and corrected Fokker-Plank type equations for the magnetic correlations. In these equations, we also incorporate the effects of ambipolar drift which would obtain if the turbulent medium has a significant neutral component. We apply these equations to discuss a number of astrophysically interesting problems: (a) the small scale dynamo in galactic turbulence with a model Kolmogorov spectrum, incorporating the effect of ambipolar drift; (b) current helicity dynamics and the quasilinear corrections to the alpha effect; (c) growth of the current helicity and large-scale magnetic fields due to nonlinear effects.

Subject headings: Magnetic fields; turbulence; dynamo processes; ambipolar drift; helicity; Galaxies

1. Introduction

The origin of large-scale cosmic magnetic fields remains at present, a challenging problem. In a standard paradigm, one invokes the dynamo action involving helical turbulence and rotational shear, to generate magnetic fields ordered on scales much larger than the turbulence scale (cf. Moffat 1978, Parker 1979, Zeldovich *et al.* 1983). However, turbulent motions, with a large enough magnetic Reynolds number (MRN henceforth), can also excite a small-scale dynamo, which exponentiates fields correlated on the tubulent eddy scale, at a rate much faster than the mean field growth rate (Kazantsev 1968, Zeldovich *et al.* 1983 and see below). A possible worry is that, these small-scale fields can come to equipartition with the turbulence much before the large-scale

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field has grown appreciably and may then interfere with the large-scale dynamo action (cf. Kulsrud and Anderson, 1992 (KA)). Indeed the efficiency of the dynamo to produce the observed large-scale field has come under increasing scrutiny (Cattaneo and Vainshtein 1991, Vainshtein and Rosner 1991, Kulsrud and Anderson 1992, Brandenburg 1994, Field 1996, Blackman 1996, Chandran 1997, Subramanian 1997). It appears therefore that understanding large-scale dynamo action due to helical turbulence is also closely linked with understanding the dynamics of the small-scale fluctuating component of the magnetic field. This forms the prime motivation for the present work, where we revist the dynamics of the fluctuating field ab initio.

Fluctuating field dynamics is best studied in terms of the dynamics of magnetic correlation functions (see below). Kazantsev (1968) derived the equations for the longitudinal correlations in homogeneous, isotropic , Markovian turbulence, without mean helicity, using a diagrammatic approach. Vainshtein and Kitchatinov (1986) incorporated the effects of helicity, and derived equations for both helical and longitudinal correlations. They did not however give details of the algebra. We therefore first rederive these equations here, in a fairly transparent manner. Our work generalises the Kazantsev equations to helical turbulence, and corrects a sign error of Vainshtein and Kitchatinov (1986) in the helical terms, which could be important for understanding the back reaction of the small-scale field on the alpha effect (see below).

The above mentioned works on small-scale field dynamics were also purely kinematic. As the field grows one expects the back reaction due to the Lorentz force to become important. The full MHD problem, involving both the Euler and induction equation, presents a formidable challenge, which we do not take up here. However in order to get a first look at the effect of nonlinearities, we consider a simpler nonlinear modification to the velocity field due to the generated magnetic field, as would obtain for example in a partially ionised medium. Note that the gas in galaxies, proto stellar disks and in the universe just after recombination are all likely to be only partially ionised. In such a medium, the Lorentz force on the charged component will cause a slippage between it and the neutrals. Its magnitude is determined by the balance between the Lorentz force and the ion-neutral collisions. This drift, called ambipolar drift (Mestel and Spitzer 1956, Draine 1986, Zweibel 1988), can be incorporated as a field dependent addition to the fluid velocity in the induction equation. This makes the induction equation nonlinear and provides us with a model nonlinear problem to study. Such a modification of the velocity field has also been used by Pouquet *et al.* (1976) and Zeldovich *et al.* (1983) (pg. 183) to discuss nonlinear modifications to the alpha effect.

We incorporate the effects of this nonlinearity while deriving the evolution equations for both the mean field and fluctuating field correlations, in section 3 and 4. The presence of a nonlinear term due to ambipolar type drift in the induction equation implies that lower moments couple to higher order moments. Some form of closure has to be assumed. We adopt here a Gaussian closure. The mean field and the magnetic field correlations then obey a set of nonlinear partial differential equations with the nonlinearity appearing as time-dependent co-efficients involving the average properties of the fluctuating field itself. In the sections which follow we apply these

equations to study a number of astrophysically interesting problems.

We begin in section 5 by examining small-scale dynamo action in the galactic context. This problem was studied by KA who looked at the evolution of the magnetic energy spectrum in wavenumber space, taking the limit of large k (small scales). In contrast to their treatment, the co-ordinate space approach adopted here, allows us to implement the boundary conditions for the magnetic correlation function, at both large and small-scales. Using the WKBJ approximation, we derive conditions for growth of small-scale fields in a model Kolmogorov turbulence and study the eigen functions for the longitudinal magnetic correlation (see below). Most earlier work also discussed small scale dynamo action for the case when the turbulent velocity has a single scale (cf. Kleeorin *et al.* 1986, Ruzmaikin *et al.* 1989 and references therein), Our work extends this to the context where multiple scales are present.

The nonlinear effects of ambipolar drift, on the small-scale dynamo are considered in section 6. The results of section 5 and 6, suggests a useful visualisation of the spatial distribution of the small-scale dynamo generated field (cf. Zeldovich et. al, 1983). We argue that in the galactic context, the magnetic field generated by small scale dynamo action, concentrates into thin (perhaps ropy) structures. One of the aims of our work here is to lay a framework for a companion paper (Subramanian, 1997 : Paper I). In Paper I we have built upon the results obtained in sections 5 and 6, to discuss in detail how the small-scale dynamo may saturate in the galactic context, in a manner which preserves large-scale dynamo action.

Section 7 concentrates on the kinematic evolution of the helical component of the magnetic correlations. The study of helical magnetic correlations has been mostly neglected in the literature, since they do not drastically affect the small-scale dynamo. However the evolution of current helicity is important in determining the back reaction of small-scale fields on the α-effect in the standard dynamo equation (cf. Pouquet *et al.* 1976, Zeldovich *et al.* 1983, Gruzinov and Diamond 1994, Bhattacharjee and Yuan 1995). Numerical simulations by Tao *et al.*, 1993 indicated that the alpha effect is drastically decreased by the growing small-scale field (see however Brandenburg 1994). We will examine this issue in terms of our approach.

In section 8, we study the nonlinear effects due to the current helicity. We point out that the current helicity may increase temporarily due to its non linear coupling with the logitudinal magnetic correlation function. We also comment on the possibility of a non linear dynamo driven by the magnetic alpha effect and on the ordering of small scale fields due to relaxation and selective decay. The final section presents a summary of the results.

2. Mathematical preliminaries

In a partially ionised medium the magnetic field evolution is governed by the induction equation

$$(\partial \mathbf{B}/\partial t) = \nabla \times (\mathbf{v}_i \times \mathbf{B} - \eta \nabla \times \mathbf{B}), \qquad (1)$$

where $\mathbf{B}$ is the magnetic field, $\mathbf{v}_i$ the velocity of the ionic component of the fluid and η the ohmic resistivity. The ions experience the Lorentz force due to the magnetic field. This will cause them to drift with respect to the neutral component of the fluid. If the ion-neutral collisions are rapid enough, one can assume that the Lorentz force on the ions is balanced by their friction with the neutrals. Under this approximation the Euler equation for the ions reduces to :

$$\rho_i \nu_{in}(\mathbf{v}_i - \mathbf{v}_n) \equiv \rho_i \nu_{in} \mathbf{v}_D = [(\nabla \times \mathbf{B}) \times \mathbf{B}]/(4\pi), \tag{2}$$

where ρ_i is the mass density of ions, ν_{in} the ion-neutral collision frequency and $\mathbf{v}_n$ the velocity of the neutral particles. We have also defined here $\mathbf{v}_D$, the ambipolar drift velocity.

Using the Euler equation for the ions and substituting for $\mathbf{v}_i$, the induction equation becomes the nonlinear equation

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times \left[\mathbf{v}_n \times \mathbf{B} + a[((\nabla \times \mathbf{B}) \times \mathbf{B}) \times \mathbf{B}] - \eta \nabla \times \mathbf{B}\right], \tag{3}$$

where we have defined

$$a = \frac{1}{4\pi \rho_i \nu_{in}} \tag{4}$$

We have derived the above equation as one which describes the effect of ambipolar drift. However, one can also view Eq. (3) as describing a model nonlinear problem, where the nonlinear effects of the Lorentz force are taken into account as simple modification of the velocity field. Such a phenemenological modification of the velocity field has in fact been used by Pouquet *et al.* (1976) and Zeldovich *et al.* (1983) (pg. 183) to discuss nonlinear modifications to the alpha effect.

The velocity field $\mathbf{v}_n$ is taken to be prescribed independent of the magnetic field. We will assume $\mathbf{v}_n$ has a turbulent stochastic component $\mathbf{v}_T$ over and above a smooth component $\mathbf{v}_0$, that is $\mathbf{v}_n = \mathbf{v}_0 + \mathbf{v}_T$. Since $\mathbf{v}_T$ is stochastic, Eq. (3) becomes a stochastic partial differential equation. Its solution depends on the statistical properties of the velocity field $\mathbf{v}_T$.

We assume $\mathbf{v}_T$ to be an isotropic, homogeneous, Gaussian random velocity field with zero mean. For simplicity, in this work, we also assume $\mathbf{v}_T$ to have a delta function correlation in time (Markovian approximation) and its two point correlation to be specified as

$$< v_T^i(\mathbf{x},t) v_T^j(\mathbf{y},s) >= T^{ij}(r)\delta(t-s) \tag{5}$$

with

$$T^{ij}(r) = T_{NN}[\delta^{ij} - (\frac{r^i r^j}{r^2})] + T_{LL}(\frac{r^i r^j}{r^2}) + C\epsilon_{ijf} r^f. \tag{6}$$

Here $<>$ denotes averaging over an ensemble of the stochastic velocity field $\mathbf{v}_T$, $r = |\mathbf{x} - \mathbf{y}|$, $r^i = x^i - y^i$ and we have written $T^{ij}(r)$ in the form appropriate for a statistically isotropic and homogeneous tensor (cf. Landau & Lifshitz 1987). $T_{LL}(r)$ and $T_{NN}(r)$ are the longitudinal and transverse correlation functions for the velocity field while $C(r)$ represents the helical part of the velocity correllations. If $\mathbf{v}_T$ is assumed to be divergence free (which we do here), then

$$T_{NN} = \frac{1}{2r}\frac{\partial}{\partial r}(r^2 T_{LL}(r)). \tag{7}$$

The stochastic Eq. (3) can now be converted into equations for the various moments of the magnetic field. To derive these equations we proceed as follows: (see also Zeldovich *et al.* 1983, Chapter 8)

3. Mean field evolution

Let the magnetic field at an initial time say $t = 0$ be $\mathbf{B}(\mathbf{x}, 0)$. Then, at an infinitesimal time δt later, the field is given iteratively by

$$\mathbf{B}(\mathbf{x}, \delta t) = \mathbf{B}(\mathbf{x}, 0) + \delta t\ \eta \nabla^2 \mathbf{B}(\mathbf{x}, 0) + \int_0^{\delta t} dt \nabla \times \mathbf{E}_A(\mathbf{x}, t) + \int_0^{\delta t} dt \int_0^{\delta t} ds \nabla \times \mathbf{E}_B(\mathbf{x}, t, s) \tag{8}$$

where we have defined $\mathbf{E}_A(\mathbf{x}, t) = \mathbf{V}(\mathbf{x}, t) \times \mathbf{B}(\mathbf{x}, 0)$, $\mathbf{E}_B(\mathbf{x}, t, s) = \mathbf{v}_T(\mathbf{x}, t) \times [\nabla \times (\mathbf{v}_T(\mathbf{x}, s) \times \mathbf{B}(\mathbf{x}, 0))]$ and $\mathbf{V} = \mathbf{v}_n + \mathbf{v}_D$. We have also retained only terms which are potentially of order δt, and so will survive in the limit $\delta t \rightarrow 0$. Note that the last term in the above equation is potentially first order in δt, because of the presence of the stocahstic turbulent velocity field $\mathbf{v}_T$.

The time evolution of the mean magnetic field $\mathbf{B}_0 =< \mathbf{B} >$ can be deduced by taking the ensemble average of Eq. (8) . As before $<>$ denotes averaging over an ensemble of the stochastic velocity field $\mathbf{v}_T$. On using the fact that $\mathbf{v}_T$ at time t is not correlated with either the initial magnetic field $\mathbf{B}(\mathbf{x}, 0)$ or the initial perturbed field $\delta\mathbf{B}(\mathbf{x}, 0) = \mathbf{B} - < \mathbf{B} >$ and taking the limit $\delta t \rightarrow 0$ we get after some straight forward algebra

$$\frac{\partial \mathbf{B}_0}{\partial t} = \nabla \times [\mathbf{v}_0 \times \mathbf{B}_0 + 2C(0)\mathbf{B}_0 - (\eta + T_{LL}(0))\nabla \times \mathbf{B}_0] + < \nabla \times \mathbf{v}_D \times \mathbf{B} > \tag{9}$$

The effect of the turbulent velocity is to introduce the standard extra terms representing the α-effect with

$$\alpha = 2C(0) = -\frac{1}{3}\int < \mathbf{v}_T(\mathbf{x}, t).\nabla \times \mathbf{v}_T(\mathbf{x}, s) > ds \tag{10}$$

and an extra turbulent contribution to the diffusion

$$\eta_T = T_{LL}(0) = \frac{1}{3}\int < \mathbf{v}_T(\mathbf{x}, t).\mathbf{v}_T(\mathbf{x}, s) > ds. \tag{11}$$

Over and above these terms the effect of ambipolar drift is to introduce an extra EMF represented by the last term in Eq. (9) . This extra term involves the third moment of the magnetic field. Similarly as we will see below a consideration of the equation for the magnetic correlation function will introduce the fourth moment of the magnetic field. This will lead to an infinite hierarchy of equations for the moments which can be only truncated by assuming some form of closure. We assume below, for analytic tractability, that the magnetic field fluctuation $\delta\mathbf{B} = \mathbf{B} - \mathbf{B}_0$ is also a homogeneous, isotropic, Gaussian random field with zero mean. Its equal time two point correlation is given by

$$< \delta B^i(\mathbf{x}, t)\delta B^j(\mathbf{y}, t) >= M^{ij}(r, t) = M_N[\delta^{ij} - (\frac{r^i r^j}{r^2})] + M_L(\frac{r^i r^j}{r^2}) + H\epsilon_{ijf}r^f. \tag{12}$$

(Here the averaging is a double ensemble average over both the stochastic velocity and stochastic $\delta\mathbf{B}$ fields, although we indicate only one angular bracket). The functions $M_L(r,t)$ and $M_N(r,t)$ are the longitudinal and transverse correlation functions for the magnetic field while $H(r,t)$ represents the helical part of the correlations. Note that H is proportional to the current-field correlation, rather than the field-vector potential correlation (see below). Since $\nabla.\mathbf{B} = 0$, M_N and M_L are related by

$$M_N = \frac{1}{2r}\frac{\partial}{\partial r}(r^2 M_L(r)). \tag{13}$$

The nonlinear term due to ambipolar drift, $< \nabla \times \mathbf{v}_D \times \mathbf{B} >$ in Eq. (9) is then given by

$$\nabla \times [\frac{2a}{3} < \delta\mathbf{B}.\nabla \times \delta\mathbf{B} > + a(\mathbf{B}_0.\nabla \times \mathbf{B}_0)]\mathbf{B}_0 - [\frac{2a}{3} < \delta\mathbf{B}.\delta\mathbf{B} > + a(\mathbf{B}_0^2)]\nabla \times \mathbf{B}_0. \tag{14}$$

Using the form for the magnetic correlation function we have

$$< \delta\mathbf{B}.\nabla \times \delta\mathbf{B} >= -6H(0,t) \text{ and } < \delta\mathbf{B}.\delta\mathbf{B} >= 3M_L(0,t). \tag{15}$$

So the mean magnetic field satisfies the equation

$$\frac{\partial \mathbf{B}_0}{\partial t} = \nabla \times [\mathbf{v}_0 \times \mathbf{B}_0 + \alpha_{eff}\mathbf{B}_0 - \eta_{eff}\nabla \times \mathbf{B}_0]. \tag{16}$$

where

$$\alpha_{eff} = 2C(0) - 4aH(0,t) + a(\mathbf{B}_0.\nabla \times \mathbf{B}_0) \tag{17}$$

$$\eta_{eff} = \eta + T_{LL}(0) + 2aM_L(0,t) + a\mathbf{B}_0^2 \tag{18}$$

The effect of ambipolar drift (or the field dependent addition to the fluid velocity) on the evolution of the mean field is therefore to modify the α-effect and the diffusion of the mean field. When one starts from small seed fields, the additional nonlinear terms depending on the mean field itself are subdominant to the terms depending on the fluctuating field for most part of the evolution; since we will find below that the small scale fields grow much more rapidly compared to the large-scale mean field. When ambipolar drift is taken into account, the small-scale fluctuating fields contribute an extra diffusion term to the mean field evolution, proportional to their energy density. Also the alpha effect is modified by the addition of a term proportional to the mean current aligned component (or current helicity, $H(0,t)$) of the magnetic field fluctuations. Some aspects of mean field dynamos incorporating ambipolar drift has been discussed by Zweibel (1988) and Proctor and Zweibel (1992).

4. Evolution of the correlation tensor of magnetic fluctuations

The derivation of these equations involves straightforward but rather tedious algebra. We therefore only outline the steps and the approximations below leaving out most of the algebraic

details. We start by noting that

$$\begin{aligned}(\partial M_{ij}/\partial t) &= (\partial/\partial t)(<\delta B_i(\mathbf{x},t)\delta B_j(\mathbf{y},t)>) \\ &= [(\partial/\partial t)(<B_iB_j>) - (\partial/\partial t)(<B_i><B_j>)]. \end{aligned} \tag{19}$$

The second term in the square brackets is easy to evaluate using the equation for the mean field. The first term can be evaluated using Eq. (3) and the fact that

$$(\partial/\partial t)(B_i(\mathbf{x},t)B_j(\mathbf{y},t)) = B_i(\mathbf{x},t)(\partial B_j(\mathbf{y},t)/\partial t) + (\partial B_i(\mathbf{x},t)/\partial t)B_j(\mathbf{y},t). \tag{20}$$

The resulting equation can again be solved iteratively to get an equation for $(\partial M_{ij}/\partial t)$ which depends on the the turbulent velocity correlations T_{ij}, the mean velocity field $\mathbf{v}_0$ and the mean magnetic field $\mathbf{B}_0$ and most importantly a non-liner term incorporating the effects of ambipolar drift. We get

$$\begin{aligned}\frac{\partial M_{ij}}{\partial t} &= <\int {}^yR_{jpq}\left[v_T^p(\mathbf{y},t)\ {}^xR_{ilm}(v_T^l(\mathbf{x},s)[M_{mq}+B_0^m(\mathbf{x})B_0^q(\mathbf{y}))\right] ds> \\ &+ <\int {}^xR_{ipq}\left[v_T^p(\mathbf{x},t)\ {}^yR_{jlm}(v_T^l(\mathbf{y},s)[M_{qm}+B_0^q(\mathbf{x})B_0^m(\mathbf{y}))\right] ds> \\ &+ <\int {}^yR_{jpq}\left(v_T^p(\mathbf{y},t)\ {}^yR_{qlm}(v_T^l(\mathbf{y},s)M_{im})\right) ds> \\ &+ <\int {}^xR_{ipq}\left(v_T^p(\mathbf{x},t)\ {}^xR_{qlm}(v_T^l(\mathbf{x},s)M_{mj})\right) ds> \\ &+ \eta[\nabla_y^2 M_{ij} + \nabla_x^2 M_{ij}] +^y R_{jpq}\,(v_0^p(\mathbf{y})M_{iq}) +^x R_{ipq}\,(v_0^p(\mathbf{x})M_{qj}) \\ &+ {}^yR_{jpq}\,(<v_D^p(\mathbf{y})\delta B_i(\mathbf{x})B_q(\mathbf{y})>) +^x R_{ipq}\,(<v_D^p(\mathbf{x})B_q(\mathbf{x})\delta B_j(\mathbf{y})>)\end{aligned} \tag{21}$$

where we have defined the operators

$${}^xR_{ipq} = \epsilon_{ilm}\epsilon_{mpq}(\partial/\partial x^l) \text{ and } {}^yR_{ipq} = \epsilon_{ilm}\epsilon_{mpq}(\partial/\partial y^l). \tag{22}$$

The first two terms on the RHS of Eq. (21) represent the effect of velocity correlations on the magnetic fluctuations (M_{ij}) and the mean field (B_0^i). The next two terms the "turbulent transport" of the magnetic fluctuations by the turbulent velocity, the 5th and 6th terms the "microscopic diffusion". The 7th and 8th terms the transport of the magnetic fluctuations by the mean velocity. The last two nonlinear terms give the effects of the back reaction due to ambipolar drift on the magnetic fluctuations.

We note that the effects of the mean velocity and magnetic fields are generally subdominant to the effects of the fluctuating velocity and magnetic fields. First the mean fields vary in general on a scale much larger than the fluctuating fields. If one neglects the effect of velocity shear due to the mean velocity, on the fluctating field, then one can transform away the mean velocity by going to a different reference frame. The mean magnetic field, ofcourse cannot be transformed away. But we will see that, it grows at a rate much slower than the fluctuating field. So when one starts from small seed magnetic fields, for most part of the evolution, we can neglect its effects. In

what follows we will drop the term involving the mean fields. Due to the above reasons, we also continue to treat the statistical properties of the magnetic fluctuations as being homogeneous and isotropic, and use $M_{ij}(\mathbf{x},\mathbf{y},t) = M_{ij}(r,t)$ as before.

All the terms in the above equation, can be further simplified by using the properties of the magnetic and velocity correlation functions. In order to obtain equations for M_L and H, we multiply Eq. (21) by $(r^i r^j)/r^2$ and $\epsilon_{ijf} r^f$ and use the identities

$$M_L(r) = M_{ij}(r^i r^j/r^2), \;\; H(r) = M_{ij}\epsilon_{ijf} r^f/(2r^2). \tag{23}$$

We have given some steps in simplifying the first two terms in appendix A. The 3rd and 4th terms add to give a contribution $4C(0)\epsilon_{jqm}(\partial M_{im}/\partial r^q) + 2T_{LL}(0)\nabla^2 M_{ij}$ to the RHS of Eq. (21) , hence justifying their being called "turbulent transport" of M_{ij}.

The last two nonlinear terms give the effects of the back reaction due to ambipolar drift on the magnetic fluctuations. In evaluating this term, we will neglect the effects of the subdominant mean field compared to the back reaction effects of the fluctuating field. In this case the nonlinear terms add to give a contribution $-8aH(0,t)\epsilon_{jqm}(\partial M_{im}/\partial r^q) + 4aM_L(0,t)\nabla^2 M_{ij}$ to the RHS of Eq. (21) . The Gaussian assumption of the magnetic correlations results in the nonlinearity of this term appearing as a nonlinearity in the coefficient, rather than the correlation function itself. Gathering together all the terms, we get for the coupled evolution equations for M_L and H :

$$\frac{\partial M_L}{\partial t} = \frac{2}{r^4}\frac{\partial}{\partial r}(r^4 \kappa_N \frac{\partial M_L}{\partial r}) + GM_L - 4\alpha_N H \tag{24}$$

$$\frac{\partial H}{\partial t} = \frac{1}{r^4}\frac{\partial}{\partial r}\left(r^4 \frac{\partial}{\partial r}(2\kappa_N H + \alpha_N M_L)\right) \tag{25}$$

where we have defined

$$\begin{aligned} \kappa_N &= \eta + T_{LL}(0) - T_{LL}(r) + 2aM_L(0,t) \\ \alpha_N &= 2C(0) - 2C(r) - 4aH(0,t) \\ G &= -4\left[\frac{d}{dr}(\frac{T_{NN}}{r}) + \frac{1}{r^2}\frac{d}{dr}(rT_{LL})\right] \end{aligned} \tag{26}$$

These equations together with Eq. (16) for the mean magnetic field are an important result of this work. They form a closed set of nonlinear partial differential equations for the evolution of both the mean magnetic field and the magnetic fluctuations, incorporating the back reaction effects of ambipolar drift (or a magnetic field dependent addition to the velocity). For non-helical turbulence the equation for M_L excluding nonlinear effects was first derived by Kazantsev (1968). We note that Eq. (24) and Eq. (25) without the inclusion of the non linear terms due to ambipolar drift, have been derived in a different fashion by Vainshtein and Kichatinov (1986). We get exactly their equation (27) for M_L and H, except for a sign difference in front of the α_N terms. We believe that our equations have the correct sign (see also below).

The terms involving κ_N in equations (24) and (25) represent the effects of diffusion on the magnetic correlations. The diffusion coefficient includes the effects of microscopic diffusion (η) and a scale-dependent turbulent diffusion ($T_{LL}(0) - T_{LL}(r)$). The effect of ambipolar drift, under our approximation of Gaussian magnetic correlations, is to add to the diffusion coefficient an amount $2aM_L(0,t)$; a term proportional to the energy density in the fluctuating fields. Similarly α_N represents first a scale dependent α-effect ($2C(0) - 2C(r)$) and the effect of ambipolar drift is to decrease this by $4aH(0,t)$, an amount proportional to the mean current helicity of the magnetic fluctuations. Ambipolar drift has therefore very similar effect on the magnetic fluctuations as on the mean field. The addition of these terms makes equations (24) and (25) nonlinear, with the non linearity appearing in the coefficients. The term proportional to $G(r)$, allows for for the rapid generation of magnetic fluctuations by velocity shear and the existence of a small-scale dynamo *independent* of the large-scale field (cf. also KA, Vainshtein and Kichatinov 1986).

An important facet of MHD equations is the conservation of magnetic helicity, in the absense of microscopic diffusion. Magnetic helicity is defined as

$$I_M = \int \mathbf{A}.\mathbf{B}\ d^3\mathbf{x} \tag{27}$$

where $\mathbf{A}$ is the vector potential. One can show, (cf. Moffat 1978) I_M is conserved in the ideal limit. In case we have no mean fields, we also have $I_M = \mathcal{V} < \mathbf{A}.\mathbf{B} >$ and

$$\frac{1}{\mathcal{V}}\frac{dI_M}{dt} = -2\eta < \mathbf{B}.(\nabla\times\mathbf{B}) > . \tag{28}$$

(Note that we are assuming that integration over a large volume $\mathcal{V}$ is same as the ensemble average). It is interesting to display this conservation explicitely from our equations. This will also act as check on some part of the algebra.

For this let us go over to the vector potential representation of the fluctuating field and write $\delta\mathbf{B} = \nabla\times\delta\mathbf{A}$. We also define the equal time correlation fuction of the fluctuating component of the vector potential as $P^{ij} =< \delta A^i\delta A^j >$. Since we have taken the fluctuting magnetic field as a homogeneous, isotropic, Gaussian random field, P_{ij} will also satisfy this property. So one can write in general

$$P^{ij}(r,t) = P_N[\delta^{ij} - (\frac{r^ir^j}{r^2})] + P_L(\frac{r^ir^j}{r^2}) + P_H\epsilon_{ijf}r^f. \tag{29}$$

Here the functions $P_L(r,t)$ and $P_N(r,t)$ are the longitudinal and transverse correlation functions for the vector potential. $P_H(r,t)$ represents the helical part of the correlations and is related to the magnetic helicity of the fluctuating field, with

$$P_H(0,t) = - < \delta\mathbf{A}.\delta\mathbf{B} > /6. \tag{30}$$

So to determine the evolution of the average magnetic helicity of the fluctuating field, we have to evaluate the time evolution of $P_H(0,t)$.

One can easily relate correlation function H, representing the current-field correlation, to P_H, by using the definitions of the two quantities. We have

$$H(r,t) = -\frac{r^f}{2r^2}\epsilon_{ijf}\epsilon_{ilm}\epsilon_{jpq}P_{mq,lp} = -\frac{1}{r^4}\frac{\partial}{\partial r}\left(r^4\frac{\partial P_H}{\partial r}\right) \tag{31}$$

Substituting this relation in the evolution equation (25) for H, we get

$$\frac{1}{r^4}\frac{\partial}{\partial r}\left[r^4\frac{\partial}{\partial r}\left(\frac{\partial P_H}{\partial t} + 2\kappa_N H + \alpha_N M_L\right)\right] = 0. \tag{32}$$

Since all the correlations die off at spatial infinity the integral of this equation then gives,

$$\frac{\partial P_H(r,t)}{\partial t} = -(2\kappa_N H + \alpha_N M_L) \tag{33}$$

Now suppose we look at the evolution of the mean helicity of the fluctuating fields, in the absense of the mean magnetic field. Then conservation of I_M implies conservation of $P_H(0,t)$. To see if this conservation obtains, take the $r \to 0$ limit of the Eq. (33). We get

$$\frac{\partial P_H(0,t)}{\partial t} = -2\eta H(0,t) \tag{34}$$

So when $\eta = 0$, the above equation shows that $P_H(0,t) =$ constant, independent of time, as required. Also when $\eta \neq 0$, and the mean fields are zero, then Eq. (34) is identical to Eq. (28) , as required. It is heartening to note that the nonlinear equation that we have derived for the helical part of the magnetic correlations, incorporating the effects of ambipolar type drifts, does indeed embody magnetic helicity conservation in the appropriate limit, as required.

Note that if the signs in front of the α_N term had been different from the one we get, then the ambipolar drift terms in the equation for P_H would not have cancelled out when we take the $r \to 0$ limit in Eq. (33). And we could not recover magnetic helicity conservation as above. This provides another check that we have indeed got the relative sign of these terms right. Also from the same argument, we note that one cannot have non-zero additions to the alpha term in the $r \to 0$ limit (the $-4aH(0,t)$ term in α_N) without a corresponding addition to the diffusion term (the $2aM_L(0,t)$ term in κ_N).

We now consider me of the implications of these equations for the evolution of small-scale fields.

5. Kinematic evolution of M_L and the small-scale dynamo

We begin by first studying the kinematic evolution of M_L, ignoring the effects of the nonlinear coupling terms due to ambipolar drift. In particular we consider the small-scale dynamo in galactic turbulence with a model Kolmogorov spectrum. The results obtained in this section will be used

to set the framework for Paper I. It also extends the analysis of the small-scale dynamo, due to turbulence with a single scale (cf. Zeldovich *et al.* (1983), Kleeorin *et al.* (1986)) to the situation when a range of scales are present.

First we note that in many contexts the coupling term $\alpha_N H$, due to helicity fluctuations of the velocity, has negligible influence on the evolution of M_L. A canonical estimate in the galactic context (see Zeldovich et al. 1983) is $\alpha_N \sim 2C(0) \sim (V^2\tau/h)(\Omega\tau) \sim \Omega\tau V(L/h)$ and $\kappa_N \sim T_{LL}(0) \sim V^2\tau \sim VL$. Here where h is the disk scale height, Ω the rotation frequency, V, τ and L are the velocity, correlation time, and correlation lengths for the energy carrying eddies of the turbulence. Assuming $H \sim M_L/h$, the importance of the coupling term $\alpha_N H$ in Eq. (24) to the other terms is $\sim (\alpha_N H/(\kappa_N M_L/L^2)) \sim \Omega\tau(L/h)^2 << 1$ in general. (Even for $\tau \sim L/V$ we generally have $\Omega\tau << 1$). So it is an excellent approximation to neglect the coupling to H in examining evolution of M_L. We will refine this estimate in the next section after studying the evolution of H and see that the above approximation is even better.

The evolution for M_L can then be transformed into a Schrodinger-type equation by defining $\Psi = r^2\sqrt{\kappa_N}M_L$. We get

$$\frac{1}{2}\frac{\partial\Psi}{\partial t} = \kappa_N\frac{\partial^2\Psi}{\partial r^2} - U(r,t)\Psi \tag{35}$$

where for a divergence free velocity field, the "potential"

$$U(r,t) = T''_{LL} + \frac{2}{r}T'_{LL} + \frac{\kappa''_N}{2} - \frac{(\kappa'_N)^2}{4\kappa_N} + \frac{2\kappa_N}{r^2}. \tag{36}$$

The boundary condition on $M_L(r,t)$ is that it be regular at the origin and that $M_L(r,t) \to 0$ as $r \to \infty$.

In the kinematic limit, note that κ_N, T_{LL} and hence the potential U are time independent. Equation (35) then admits eigenmode solutions of the form $\Psi(r,t) = \exp(2\Gamma t)\Phi(r)$ where

$$\kappa_N(d^2\Phi/dr^2) - (\Gamma + U)\Phi = 0. \tag{37}$$

So there exists a possibility of growing modes with $\Gamma > 0$, if one can have U sufficiently negative in some range of r. The problem of having a small-scale dynamo and rapid growth of magnetic fluctuations, reduces to that of having bound states in the potential U.

To see what this requires let us consider a model problem where the behaviour $T_{LL}(r)$ simulates Kolmogorov turbulence (Vainshtein 1982);

$$\begin{aligned} T_{LL}(r) &= \frac{VL}{3}[1 - R_e^{1/2}(\frac{r}{L})^2] \quad \text{for } 0 < r < l_c \\ &= \frac{VL}{3}[1 - (\frac{r}{L})^{4/3}] \quad \text{for } l_c < r < L \\ &= 0 \qquad \text{for } r > L \end{aligned} \tag{38}$$

Here $l_c \approx LR_e^{-3/4}$ is the cut off scale of the turbulence, where $R_e = VL/\nu$ is the fluid Reynolds number and ν is the kinematic viscosity. For Kolmogorov turbulence, the eddy velocity at any

scale l, is $v_l \propto l^{1/3}$, in the inertial range. So the scale dependent diffusion coefficient scales as $v_l l \propto l^{4/3}$. This scaling, also referred to as Richardsan's law, is the motivation for the form of the scaling of T_{LL} with r which we have adopted for the inertial range of the turbulence. Note that the structure function $T_{LL}(r)$ must satisfy the condition $T'_{LL}(0) = 0$, at the origin. So for scales smaller than l_c we have continued T_{LL} from its value at $r = l_c$ to zero, satisfying this constraint. (One can adopt smoother continuations for $T_{LL}(r)$ at $r = l_c$ and $r = L$, in order to make the potential U continuous at these points. But this has little effect on the conclusions below, since Φ is determined by integrals over U).

The potential is then $U = (2\eta/r^2)$ as $r \to 0$ and $U = 2(\eta + \eta_T)/r^2$ for $r > L$. For $l_c < r < L$ we have

$$U = \frac{V}{3L}\left[-\frac{8}{9}(\frac{r}{L})^{-2/3} - \frac{(4/9)(r/L)^{2/3}}{(3/R_m + (r/L)^{4/3})} + \frac{6}{R_m}(\frac{L^2}{r^2})\right], \tag{39}$$

where $R_m = (VL/\eta)$ is the magnetic reynolds number at the outer scale of the turbulence. The Spitzer value of the resistivity gives $\eta = 10^7(T/10^4K)^{-3/2}cm^2s^{-1}$. For numerical estimates we generally take $V = 10kms^{-1}$ and $L = 100pc$. For these turbulence parameters $R_m = 3 \times 10^{19}$.

The value of the potential at $r = L$ is $U \approx (V/L)[(2/R_m) - (4/9)]$. If $R_m < 9/2$ one can easily see from the above expressions that U remains positive for all r. Note that for a bound state to obtain and for the small-scale fields to grow, one must have U negative for some range of r. So a neccessary condition for the small scale fields to grow is $R_m >> 1$. In fact the exact critical MRN, say R_c, for growth has to be determined numerically but one typically gets $R_c \sim 60$ (cf., Zeldovich et al 1983, Novikov et al. 1983; see below for a WKBJ derivation of this limit)

Since the velocity at any scale l, say $v_l \propto l^{1/3}$ for Kolmogorov turbulence, the MRN associated with eddies of scale l, is $R_m(l) = v_l l/\eta = R_m(l/L)^{4/3}$. Using this one can also rewrite the potential U as

$$U = \frac{v_l}{3l}\left[-\frac{8}{9}(\frac{r}{l})^{-2/3} - \frac{(4/9)(r/l)^{2/3}}{(3/R_m(l) + (r/l)^{4/3})} + \frac{6}{R_m(l)}(\frac{L^2}{r^2})\right]. \tag{40}$$

This is exactly of the same form as Eq. (39) except that L, V and R_m are replaced by l, v_l and $R_m(l)$ respectively. So a number of conclusions about the generation of small-scale fields can be scaled to apply to an arbitrary scale, l, provided we use the corresponding velocity scale v_l and Reynolds number $R_m(l)$ appropriate to the scale l. For example, the condition for excitation of small-scale dynamo modes which are concentrated at a scale l, is also $R_m(l) = R_c >> 1$. Of course, exact scalability of the results only obtains well in the inertial range, far away from l_c and L. The MRN associated with eddies at the cut off scale is $R_m(l_c) = v_c l_c/\eta = V(l_c/L)^{1/3}L(l_c/L)/\eta = R_m/R_e$. (Here v_c is the eddy velocity at the cut-off scale). So if $R_m/R_e >> 1$, a potential well with U negative, extends up to the cut off scale of the turbulence, and these modes will also grow.

We will be considering a largely neutral galactic gas and for this ν is dominated by the neutral contribution. We take the neutral-neutral collision to be dominated by H-H collisons with a cross section $\sigma_{H-H} \sim 10^{-16}cm^{-2}$, leading to a kinematic viscosity $\nu \sim v_{th}(1/n_H\sigma_{H-H})$. For a thermal

velocity $v_{th} \sim 10kms^{-1}$ and a neutral hydrogen number density $n_H \sim 1cm^{-3}$, as say appropriate for a young galaxy, we have $\nu \sim 10^{22}cm^2s^{-1}$, and

$$R_e = \frac{VL}{\nu} \approx 3 \times 10^4 V_{10} L_{100} \tag{41}$$

where $V_{10} = (V/10\mathrm{kms}^{-1})$ and $L_{100} = (L/100\mathrm{pc})$. So the condition $R_m/R_e > R_c$ in fact holds in the galactic context.

It should also be noted that the value of the potential at any l, $U(l) \sim v_l/l$, is the inverse of the turnover time of the eddies of scale l. The depth of the potential well at some scale l reflects the growth rate of modes concentrated around that l. And the growth rate of a mode extending up to $r \sim l$, say $\Gamma_l \sim U(l) \sim v_l/l \sim l^{-2/3}$, decreases with increasing l. So when $R_m(l_c) = R_m/R_e > R_c$, the small-scale fields tangled at the cut off scale grow more rapidly than any of the larger scale modes. These results have also been found in a different fashion by KA. To illustrate some of these points in a more quantitative fashion we have given a detailed a WKBJ analysis of Eq. (37) in appendix B. Below we summarise the main results of this analysis:

- The WKBJ analysis finds a critical value of the MRN, $R_m = R_c \approx 60$, for the excitation of the small-scale dynamo. Above this critical MRN the small-scale dynamo can lead to an exponential growth of the fluctuating field correlated on a scale L. We refer to the eigenmode which is excited for $R_m = R_c$ as the marginal mode. Further, the equations determining R_c are the same if we replace (L, R_m) by $(l, R_m(l))$. Therefore, the critical MRN for excitation of a mode concentrated around $r \sim l$ is also $R_m(l) = R_c$, as expected from the scale invariance in the inertial range.

In the galactic context $R_m >> R_c$; in fact, one also has $R_m(l_c) = R_m/R_e >> 1$. Hence, small-scale dynamo action excites modes correlated on all scales from the cut-off scale l_c to the external scale L of the turbulence.

- As expected, due to small-scale dynamo action, the fluctuating field, tangled on a scale l, grows exponentially on the corresponding eddy turnover time scale, with a growth rate $\Gamma_l \sim v_l/l \propto l^{-2/3}$. In the galactic context, with $R_m(l_c) = R_m/R_e >> R_c$, the small-scale fields tangled at the cutoff scale grow more rapidly than any of the large-scale modes.

- The WKBJ analysis gives a growth rate $\Gamma_c = (v_c/l_c)[5/4 - c_0(\ln(R_m/R_e))^{-2}]$ with $c_0 = \pi^2/12$ for the fastest growing mode. Note that this is only weakly (logarithmically) dependent on R_m, provided R_m is large enough.

To examine the spatial structure for various eigenmodes of the small-scale dynamo, it is more instructive to consider the function $w(r,t) =< \delta\mathbf{B}(\mathbf{x},t).\delta\mathbf{B}(\mathbf{y},t) >$, which measures the correlated

dot product of the fluctating field ($w(0) =< \delta\mathbf{B}^2 >$). Firstly there is a general constraint that can be placed on $w(r)$. Since the fluctuating field is divergence free, we have

$$w(r,t) = \frac{1}{r^2}\frac{d}{dr}\left[r^3 M_L\right], \tag{42}$$

so

$$\int_0^\infty w(r) r^2 dr = \int_0^\infty \frac{d}{dr}\left[r^3 M_L\right] = 0, \tag{43}$$

since M_L is regular at the origin and vanishes faster than r^{-3} as $r \to \infty$. Therefore the curve $r^2 w(r)$ should have zero area under it. Since $w(0) =< (\delta\mathbf{B})^2 >$, w is positive near the origin. And the fluctuating field points in the same direction for small separation. As one goes to larger values of r, there must then values of r, say $r \sim d$, where $w(r)$ becomes negative. For such values of r, the field at the origin and at a separation d are pointing in opposite directions on the average. This can be intepreted as indicating that the field lines, on the average are curved on the scale d.

- We find that, in the case $R_m/R_e >> 1$, $w(r)$ is strongly peaked within a region $r = r_d \approx l_c(R_m/R_e)^{-1/2}$ about the origin, for all the modes. Note that r_d is the diffusive scale satisfying the condition $\eta/r_d^2 \sim v_c/l_c$. For the most rapidly growing mode, $w(r)$ changes sign across $r \sim l_c$ and rapidly decays with increasing r/l_c. For slower growing modes, with $\Gamma_l \sim v_l/l$, $w(r)$ extends up to $r \sim l$ after which it decays exponentially.
- For the marginal mode $w(r)$ peaks within a radius $r \sim L/R_c^{3/4}$, changes sign to become negative at $r \sim L$ and dies rapidly for larger r/L.

We should point out that a detailed analysis of the eigenfunctions can be found in Kleeorin et. al. (1986), for the simple case when the longitudinal velocity correlation function has only a single scale. Their analysis is also applicable to the mode near the cut-off scale of Kolmogorov type turbulence. These authors also give a pictorial intepretation of the correlation function, in terms of the Zeldovich rope-dynamo (cf. Zeldovich *et al.* 1983). If one adopts this intepretation, the small-scale field can be thought as being concentrated in rope like structures with thickness r_d and curved on a scale upto $\sim l$ for a mode extending to $r \sim l$. In Paper I, we also elaborate on a qualitative picture of the mechanism for the dynamo growth of small scale fields, and the generation of ropy fields from general initial conditions.

As the small sclale fields grow the back reaction due to ambipolar drift will become important. We now turn to the nonlinear effects on the small scale dynamo to see when this can lead to small-scale dynamo saturation.

6. Non-linear effects on the small-scale dynamo

As the small-scale fields grow, ambipolar drift adds to the diffusion coefficient an amount $2aM_L(0,t)$; a term proportional to the energy density in the fluctuating fields. Similarly,

ambipolar drift leads to a decrease of α_N by $4aH(0,t)$, an amount proportional to the mean current helicity of the magnetic fluctuations. When considering the evolution of the longitudinal correlation function, we once again neglect the subdominant effect of the the coupling term $\alpha_N H$, due helicity fluctuations (see above).

As $M_L(0,t)$ grows, its effect then is simply to change η to an effective

$$\eta_{ambi} = \eta + 2aM_L(0,t) \tag{44}$$

in the expression for the potential $U(r,t)$. One can define an effective MRN, for fluid motion on any scale of the turbulence

$$R_{ambi}(l) = \frac{v_l l}{\eta_{ambi}} \approx \frac{v_l l}{2aM_L(0,t)} \tag{45}$$

where $v_l = (l/L)^{1/3}V$ as before.

As the energy density in the fluctuating field increases $R_{ambi}(l)$ decreases. Firstly, this makes it easier for the field energy to reach the diffusive scales $r_d \sim l_c/R_{ambi}^{1/2}(l_c)$, from a general initial configuration. More importantly as this happens the potential well disappears, first at small-scales and then progressively at larger and larger scales. This means that M_L will grow slower and slower. The detailed evolution of M_L will be complicated. However, the decrease of R_{ambi} suggests one possible nonlinear saturation mechanism for the small-scale field. The possibility that the system finds the stationary state with $(\partial M_L/\partial t) = 0$. In such a state, M_L is independent of time. So, the condition on the critical MRN for the stationary state to be reached, will be identical to that obtained in the kinematic stage. From the discussion on the kinematic evolution of M_L, the stationary state is an eigenmode for the system which obtains when the energy density of magnetic fluctuations has grown such that

$$R_{ambi}(L) = \frac{VL}{\eta + 2aM_L(0,t)} = R_c. \tag{46}$$

So if $R_{ambi}(L)$ decreases to a value $R_c \sim 60$, dynamo action will stop completely.

Let us consider now whether this condition can obtain in turbulent, partially ionised galactic gas. Take for example $V \sim 10$km s^{-1} and $L \sim 100$pc appropriate for galactic turbulence. Also let us assume that the galaxy had very nearly primordial composition in its early stage of evolution: then the ions are mostly just protons and the neutrals are mostly hydrogen atoms. We estimate in Paper I that $\rho_i \nu_{in} = n_i \rho_n < \sigma v >_{eff}$ with $< \sigma v >_{eff} \sim 4 \times 10^{-9} cm^3 s^{-1}$. Here n_i is the ion number density. We then have

$$R_{ambi}(l) = \frac{1}{f(l)} \frac{3\rho_i \nu_{in} l}{2\rho_n v_l} = \frac{Q(l)}{f(l)}, \tag{47}$$

where $f(l) = B_l^2/(4\pi\rho_n v_l^2)$ is the ratio of the local magnetic energy density of a flux rope curved on scale l, to the turbulent energy density $\rho_n v_l^2/2$ associated with eddies of scale l. Using the value of ν_{in} as determined above and putting in numerical values we get

$$Q(l) = \frac{3\rho_i \nu_{in} l}{2\rho_n v_l} \sim 1.8 \times 10^4 n_{-2} (\frac{l}{L})^{2/3} L_{100} V_{10}^{-1} \tag{48}$$

where $n_{-2} = (n_i/10^{-2}cm^{-3})$, and we have assumed a Kolmogorov scaling for the turbulent velocity fluctuations.

One can see from Eq. (47) - (48) that, for typical parameters associated with galactic turbulence, the MRN incorporating ambipolar drift is likely to remain much larger than R_c for most scales of the turbulence, even when the field energy density becomes comparable to the equipartition value. Only if the galactic gas is very weakly ionised, with $n_i < 10^{-5.5}\mathrm{cm}^{-3}$, can ambipolar drift by itself lead to small-scale dynamo saturation. Such small ion densities may perhaps obtain in the first collapsed objects in the universe, which collapse at relatively high redshifts (cf. Tegmark *et al.* 1997). However for most regions of a disk galaxy, the ion density is likely to be larger. So one expects the field to continue to grow rapidly, even taking into account ambipolar drift. Note also that the growth rates for the small-scale dynamo generally depends only weakly on the MRN, provided the MRN is much larger than R_c. (see section 5, and Kleeorin et al. 1986). Therefore, we still expect the small-scale dynamo-generated field to grow almost exponentially on the eddy turn around time scale, as long as $R_{ambi} >> R_c$. The spatial structure of the fluctuating field will also remain ropy, as in the kinematic regime, as long as $R_{ambi} >> R_c$.

How then does the small-scale dynamo saturate in galaxies? In Paper I we consider this question in some detail by examining other nonlinear feedback processes which could limit the growth of the galactic small-scale dynamo, taking into account also ambipolar drift. We briefly suumarise our findings here for the sake of completeness. The reader is referred to Paper I for details.

It turns out that the effect of the growing magnetic tension associated with the small-scale dynamo generated field is crucial. This tension acts to straighten out the curved flux ropes, while frictional drag damps the magnetic energy associated with the wrinkle in the rope. Also, small-scale flux loops can collapse and disappear. For a significantly neutral geas, these non-local effects turn out to operate on the eddy turnover time scale, when the peak field in a flux rope has grown to a few times the equipartition value. Their net effect is to make the random stretching needed for the small-scale dynamo inefficient and hence saturate the small-scale dynamo. However, the average energy density in the saturated small-scale field is sub equipartition, since it does not fill the volume. It is probable then that the small-scale dynamo generated fields do not drain significant energy from the turbulence, nor convert eddy motion of the turbulence on the outer scale to wave-like motion. So the diffusive effects needed for the large-scale dynamo operation can be preserved. This picture of small-scale dynamo saturation obtains only when the ion density is less than a critical value of $n_i^c \sim 0.06 - 0.5cm^{-3}(n_n/cm^{-3})^{2/3}$.

We note in passing that in the case of nearly neutral disks around protostars, the parameters could indeed be such that small-scale dynamo action saturates due to purely ambipolar drift (cf. Proctor and Zweibel 1992). We now turn to the the evolution of the current-field correlations.

7. Kinematic evolution of current helicity and α-effect suppression

The evolution of H is governed by Eq. (25) derived in section 4. We consider first the kinematic linear evolution when the nonlinear terms in κ_N and α_N are ignored. We saw above that the coupling of H to M_L is unimportant for the evolution of the longitudinal correlation function M_L, which could grow due purely to self-coupling terms. However for the evolution of H, its coupling to M_L provided by the α_N term cannot be neglected, because without the forcing by the M_L term in Eq. (25), any initial distribution of H decays with time due to diffusion. To see this multiply (25) by $2Hr^4\kappa_N$ and integrate over all r, neglecting the $\alpha_N M_L$ coupling term. We get

$$\frac{\partial}{\partial t}\left[\int_0^\infty H^2\kappa_N r^4 dr\right] = 2\kappa_N H r^4\frac{\partial}{\partial r}(2\kappa_N H)|_0^\infty - \int_0^\infty dr r^4[\frac{\partial}{\partial r}(2\kappa_N H)]^2. \tag{49}$$

Neglecting the boundary term for a sufficiently rapidly falling H we see that any smooth H will decay with time in the absence of the coupling to M_L.

So we have to consider the full inhomogeneous partial differential equation (25) for H, taking M_L to be a given function of r and t determined by the analysis of section 5. The solution to (25) can be written then as the sum of the solution to the homogeneous equation H_0 and a particular solution, say H_p, forced by the presence of M_L. H_0 decays with time, as shown above and therefore the asymptotic evolution of H in time will be governed by H_p. To determine $H_p(r,t)$ we use the Green's function technique (cf. Burton 1989). We are particularly interested in determining H_p when the longitudinal correlation function $M_L = \exp(\Gamma_n t)M_n(r)$, where the function $M_n(r)$ is an eigenmode of section 5, with growth rate Γ_n. In this case we can write $H_p(r,t) = H_n(r)\exp(\Gamma_n t)$, with $H_n(r)$ satisfying the time independent differential equation

$$\Gamma_n H_n(r) = \frac{1}{r^4}\frac{d}{dr}\left(r^4\frac{d}{dr}(2\kappa_N H_n + \alpha_N M_L)\right) \tag{50}$$

The boundary conditions are that $H_n(r)$ is regular at $r=0$ and $H_n(r)\to 0$ as $r\to\infty$, as before.

For general $\kappa_N(r)$ and $\alpha_N(r)$ there are no general solutions to (50). We obtain below approximate solution using the WKBJ approximation. To implement the boundary condition at $r=0$, under the WKBJ approximation, again it is better to transform to a new radial co-ordinate x, where $r = e^x$. Substituting $H_n(x) = f(x)\exp(-3x/2)/\kappa_N$, equation (50) then becomes

$$\frac{d^2 f}{dx^2} - \left(\frac{9}{4} + \frac{\Gamma_n e^{2x}}{2\kappa_N}\right) f = -e^{3x/2}\left(\frac{d^2(\alpha_N M_n)}{dx^2} + 3\frac{d(\alpha_N M_n)}{dx}\right) \tag{51}$$

The boundary conditions now become $f(x)\to 0$ as $x\to\pm\infty$.

The WKBJ solution of the homogeneous equation for $f = f_H$ are

$$f_H = F_\pm(x) = \left(\frac{9}{4} + \frac{\Gamma_n e^{2x}}{2\kappa_N}\right)^{-1/4} \exp(\pm\int_a^x \left(\frac{9}{4} + \frac{\Gamma_n e^{2x'}}{2\kappa_N(x')}\right)^{1/2} dx') \tag{52}$$

Note that $F_+(x)$ satisfies the left boundary condition $F_+ \to 0$ as $r = e^x \to 0$, while F_- staisfies the right boundary condition on f. The Wronskian of the two solutions is $W = F_+ d(F_-)/dx - d(F_+)/dx F_- = -2$. The particular solution to (51) , which satisfies the given boundary conditions can then be written as

$$f(x) = \int_{-\infty}^{\infty} dy G(x,y) e^{3y/2} \left(\frac{d^2(\alpha_N M_n)}{dy^2} + 3\frac{d(\alpha_N M_n}{dy} \right) \tag{53}$$

where the Green function $G(x,y)$ is given by

$$G(x,y) = -\frac{1}{W} \left[\theta(y-x) F_+(x) F_-(y) + \theta(x-y) F_+(y) F_-(x) \right]. \tag{54}$$

Here $\theta(x)$ is the standard heavyside function, which is zero for $x < 0$ and is unity for positive x.

Transforming back to the original radial co-ordinate r we finally have

$$H_p(r,t) = \frac{\exp(\Gamma_n t)}{2\kappa_N(r)} \int_0^{\infty} du (\frac{u}{r})^{3/2} \frac{1}{2} \left[\theta(u-r) K(r;u) + \theta(r-u) K(u;r) \right] \times \frac{1}{u^3} \frac{d}{du} \left(u^4 \frac{d}{du} (\alpha_N M_n(u)) \right) \tag{55}$$

where we have defined

$$K(r;u) = \left(\frac{9}{4} + \frac{\Gamma_n r^2}{2\kappa_N(r)} \right)^{-1/4} \left(\frac{9}{4} + \frac{\Gamma_n u^2}{2\kappa_N(u)} \right)^{-1/4} \exp\left[\int_u^r \left(\frac{9}{4} + \frac{\Gamma_n r'^2}{2\kappa_N(r')} \right)^{1/2} \frac{dr'}{r'} \right]. \tag{56}$$

Of primary interest is the behaviour of $H_P(0,t)$, the time evolution of the average current helicity associated with the fluctuating field, since it is this quantity which alters the alpha effect in the equation for the mean field evolution. This can be evaluated by putting $r = 0$ in equation (55) . We get

$$H_p(0,t) = \frac{\exp(\Gamma_n t)}{\sqrt{6}} \int_0^{\infty} dy \frac{\exp(-\int_0^y \frac{dy'}{y'} \left[(\frac{9}{4} + \frac{\kappa_N(0) y'^2}{\kappa_N(y')})^{1/2} - \frac{3}{2} \right]}{(\frac{9}{4} + \frac{\kappa_N(0) y^2}{\kappa_N(y)})^{1/4}} \times \frac{1}{y^3} \frac{d}{dy} \left(y^4 \frac{d}{dy} (\frac{\alpha_N M_n(y)}{2\kappa_N(0)}) \right) \tag{57}$$

where we have used the identity $(u/r)^{3/2} = \exp(\int_r^u (3dx/2x))$ and also defined new variables $y = u/a$ and $y' = r'/a$ with $a^2 = 2\kappa_N(0)/\Gamma_n$.

We can calculate $H_P(0,t)$ explicitely once the functional forms of $\alpha_N(y)$, $\kappa_N(y)$ and $M_n(y)$ are specified. For this let us consider the example of the previous section, where we $T_{LL}(r)$ is given by (38) , simulating Kolmogorov turbulence. In the previous section we saw that in the case $R_m/R_e >> 1$, all the modes were strongly peaked about a radius $r = r_d \sim l_c (R_m/R_e)^{-1/2}$. And the growth rate $\Gamma_n \sim v_l/l$ for a mode extending up to $r \sim l$ with $\Gamma_n \sim R_e^{1/2} V/L \equiv \Gamma_c$ for the fastest growing mode. As a model we therefore adopt a Gaussian form for $M_n(r)$

$$M_n(r) = M_n(0) \exp[-\frac{r^2}{2r_d^2}] = M_n(0) \exp(-by^2) \tag{58}$$

where $b = (a^2/2r_d^2) = (2\eta/\Gamma_n)(R_m/R_e l_c^2)$. Using the fact $(2\eta/\Gamma_c)(R_m/R_e l_c^2) = 1$, we have $b = (\Gamma_c/\Gamma_n) > 1$ for all modes.

Further, since $r_d << l_c$ in general, in evaluating the integral in Eq. (57) it will also suffice to take the form of $T_{LL}(r)$ in the region $0 < r < l$ to get $(\kappa_N(0)/\kappa_N(y)) = (1 + 2by^2/3)^{-1}$. Also, we can model the scale dependent $\alpha_N(r)$ as follows: At any scale $r > l_c$, in the galactic context, the helicity structure function will scale as

$$2C(0) - 2C(r) \sim (v_r^2\tau_r/h)(\Omega\tau_r) = (\Omega/h)r^2, \tag{59}$$

where v_r and τ_r are the eddy velocity and correlation time at any scale r which staisfy the relation $v_r\tau_r = r$. Also $C(r)$ is continuous at $r = l_c$, should have zero slope at the origin, and should vanish for $r > L$. A model helical correlation function which satisfies all the above requirements is given by

$$2C(r) = \frac{\Omega L^2}{h}[1 - \frac{r^2}{L^2}] \quad \text{for } 0 < r < L\,; C(r) = 0 \quad \text{for } r > L. \tag{60}$$

So in the kinematic regime we have

$$\frac{\alpha_N(y)}{2\kappa_N(0)} = \frac{\Omega}{2\kappa_N(0)h}y^2a^2 = \frac{\Omega}{\Gamma_n h}y^2. \tag{61}$$

Using the above forms of $\alpha_N(y)$, $\kappa_N(y)$ and $M_n(y)$ in Eq. (57) we get

$$H_P(0,t) = C_1(b)\frac{\Omega}{\Gamma_n h}M_n(0)\exp(\Gamma_n t) \tag{62}$$

where C_1 is given by the integral

$$C_1(b) = \frac{1}{\sqrt{6}}\int_0^\infty dy \frac{\exp(-\int_0^y \frac{dy'}{y'}\left[(\frac{9}{4} + \frac{y'^2}{1+2by'^2/3})^{1/2} - \frac{3}{2}\right]}{(\frac{9}{4} + \frac{y^2}{1+2by^2/3})^{1/4}} \times \frac{1}{y^3}\frac{d}{dy}\left(y^4\frac{d}{dy}(y^2\exp(-by^2))\right) \tag{63}$$

The constant C_1 can be comupted numerically for any b. We give below an approximate analytical estimate. First we note that the integrand is suppressed exponentially for $y >> 1$. It then suffices to expand the square root in the exponential, assuming $y^2/(1 + 2by^2/3) << 9/4$. The exponential term then gives a term $(1 + 2by^2/3)^{-1/4b}$ to the integrand of Eq. (63) . Consider first the case $b = 1$, appropriate for the fastest growing mode. After repeated integration by parts we then have

$$C_1(1) \approx \frac{5}{27}\int_0^\infty dw \frac{w^3\exp(-w^2)}{(1 + 10w^2/9)^{5/4}} + \frac{125}{243}\int_0^\infty dw \frac{w^5\exp(-w^2)}{(1 + 10w^2/9)^{9/4}} \tag{64}$$

Note firstly that $C_1(1) > 0$. Now $y^n e^{-y^2}$ is strongly peaked about $y = (n/2)^{1/2}$ for large n One can then replace the remaining slowly varying parts in the integrands above by its value at the peak. The remaining integrals are gamma functions, which can be evaluated to give $C_1 \approx 0.05$ for for the fastest growing mode.

For modes which extend up to $r \sim L$, with growth rate $\sim V/L$, $b = R_e^{1/2} \sim 170$. For $b >> 1$, $1/4b << 1$ and the contribution to the integrand in (63) from the exponetial term is $\approx (1 + 2by^2/3)^{-1/4b} \sim 1$. Also for $b >> 1$, we have main contribution to the integral coming from regions $y^2 \sim b^{-1}$; In this case an approximate evaluation of the integral along lines similar to (64) gives $C_1(b) \approx 10^{-2}b^{-2}$, for large $b >> 1$.

A number of comments are in order: Firstly, note that in the kinematic regime, $H(0,t)$ grows exponentially. So the assumption of small-scale stationarity made for example by Gruzinov and Diamond (1994) to derive a constraint on H is not valid. Also since C_1 is positive, the growth of small-scale fields and hence $H(0,t)$ goes to decrease the effective alpha effect. The extent of the decrease depends on how much the small-scale field grows before it saturates. In the kinematic regime, a mode with growth rate $\Gamma_n \sim v_l/l$, for example, leads to a reduction in the alpha effect to a value

$$\alpha_{eff} = 2C(0) - 4aH(0,t) = \frac{\Omega L^2}{h}\left[1 - \frac{4aC_1(b_n)M_n(0)e^{\Gamma_n t}}{\Gamma_n L^2}\right] = 2C(0)\left[1 - (\frac{l}{L})^2 \frac{2C_1(b_n)}{R_{ambi}(l,t)}\right]. \quad (65)$$

Since $C_1 << 1$ in general and $R_{ambi}(l) >> 1$ from Eq. (48), we see that the reduction to the α-effect due to ambipolar drift is negligible even as the magnetic field energy grows to equipartition. A similar result obtains if we use the reduction in alpha effect estimated in the quasilinear approximation of Gruzinov and Diamond (1994). In our notation their $\alpha_{eff} = \alpha_{GD} = 2C(0) - \tau H(0,t)/(\pi\rho_n)$, where τ is a turbulence correlation time. Using $H(0,t)$ determined above, one then gets for a mode with growth rate $\Gamma_n \sim v_l/l$,

$$\alpha_{GD} = 2C(0)[1 - (4C_1/3)f(l,t)(v_l\tau l)/L^2]. \quad (66)$$

Here we have assumed that the small scale field energy density is a fraction $f(l,t)$ of the energy density associated with eddies of scale l. Again for $C_1 << 1$, the correction to alpha in the quasilinear regime is modest, as long as $f(l,t) < 1$. Ofcourse, it is unclear what is the domain of validity of the quasilinear approach. One should take the estimate in Eq. (66), of alpha effect reduction at best as a rough guide, since neither we (nor Gruzinov and Diamond 1994) are not treating the full nonlinear MHD problem. Finally note that $H = C_1(\Omega/\Gamma_n)M_L/h << M_L/h$, since $(\Omega/\Gamma_n) << 1$ and $C_1 << 1$. So our neglect of the α_N coupling term for the evolution of M_L, in section 5 is an even better approximation than was argued there.

8. Non-linear effects due to current helicity

As mentioned earlier the effect of ambipolar drift is to add to the diffusion coefficient an amount $2aM_L(0,t)$ and decrease α_N by $4aH(0,t)$, in both the mean field and the correlation function equations. We now consider some implications of the additional term due to the current helicity $H(0,t)$. First let us look at the non linear evolution of $H(r,t)$ itself. The general solution

of (25) for the evolution of H, when M_L is also settling towards a saturated state is beyond the scope of this work. The following general comments can however be made: the increase in κ_N due to ambipolar diffusion will lead to a further damping of H. However the change (decrease) in α_N can lead in principle to a further increase in H due to the fact that M_L has a negative curvature near the origin.

To illustrate this more interesting nonlinear effect, we consider the following simpler problem. We assume that M_L has attained a saturated state and also ignore the effects of turbulent diffusion for analytical tractability. The equation for H then becomes

$$[\frac{\partial}{\partial t} - 2\eta_{ambi}\frac{1}{r^4}\frac{\partial}{\partial r}\left(r^4\frac{\partial}{\partial r}\right)]H = \frac{1}{r^4}\frac{\partial}{\partial r}\left(r^4\frac{\partial}{\partial r}(\alpha(r)M_L(r) - 4aH(0,t)M_L(r))\right) \equiv \rho(r,t) \qquad (67)$$

where $\eta_{ambi} = \eta + 2aM_L(0)$ as before and $\alpha(r) = 2C(0) - 2C(r)$. One can formally solve this equation using the Greens function for the operator on the LHS of Eq. (67). One gets

$$H(r,t) = \int_0^t dt' \int_0^\infty r'^4 dr' G(r,t;r',t')\rho(r',t') + \int_0^\infty r'^4 dr' G(r,t;r',0)H(r',0) \qquad (68)$$

The Greens function G can be obtained by standard methods (see for example Burton 1989); we get

$$G(r,t;r',t') = \theta(\tau)\frac{1}{(4\eta_{ambi}\tau)^{5/2}}\exp[-\frac{(r^2+r'^2)}{8\eta_{ambi}\tau}]\frac{I_{3/2}(rr'/(4\eta_{ambi}\tau))}{(rr'/(4\eta_{ambi}\tau))^{3/2}} \qquad (69)$$

where we have defined $\tau = t - t'$ and $I_{3/2}(x) = (2/(\pi x))^{1/2}[\cosh x - \sinh x/x]$ is the modified Bessel function. One can see that the Green function G is similar to that of the diffusion equation ; infact it is akin to the radial Green function for diffusion in 5-spatial dimensions. We are again particularly interested in the evolution of the current helicity $H(0,t)$. To look at its evolution take the limit $r \to 0$ in Eq. (68) to get

$$\begin{aligned} H(0,t) &= \frac{1}{3}(\frac{2}{\pi})^{1/2}\int_0^t dt' \int_0^\infty r'^4 dr' \theta(\tau)\frac{1}{(4\eta_{ambi}\tau)^{5/2}}\exp[-\frac{(r'^2)}{8\eta_{ambi}\tau}]\rho(r',t') \\ &+ \frac{1}{3}(\frac{2}{\pi})^{1/2}\int_0^\infty r'^4 dr' \frac{1}{(4\eta_{ambi}t)^{5/2}}\exp[-\frac{(r'^2)}{8\eta_{ambi}t.}]H(r',0) \end{aligned} \qquad (70)$$

An initial distribution of $H(r,0)$ will be damped by diffusion. Also since the term proportional to $\alpha(r)$ in $\rho(r,t)$ is constant in time the genetration of $H(0,t)$ due to this term will also be damped eventually by diffusion. The only potentially interesting term for the further growth of H is the nonlinear term in $\rho(r,t) \propto H(0,t)$. To see its effect, let us assume that the steady state longitudinal magnetic correlation function can be described by $M_L(r) = M_L(0)\exp[-r^2/(2L_m^2)]$. We also take an intial $H(r,0) = H(0,0)\exp[-r^2/(2L_H^2)]$. Using Eq. (70) , doing the r' integrals we then get

$$H(0,t) = \frac{20aM_L(0)}{L_m^2}\int_0^t dt'\theta(t-t')\frac{H(0,t')}{(1+4\eta_{ambi}\tau/L_m^2)^{7/2}} + \frac{H(0,0)}{(1+4\eta_{ambi}t/L_H^2)^{5/2}} \qquad (71)$$

This is an integral equation for $H(0,t)$. An approximate solution to this equation can be obtained as follows. Consider times $0 < t < t_1$ such that $4\eta_{ambi} t_1/L_m^2 < 1$. For these times one may approximate the integrand in Eq. (71) by just $H(0,t')$. Then $H(0,t)$ satisfies the simple differential equation

$$dH(0,t)/dt = 20aM_L(0)\left[\frac{H(0,t)}{L_m^2} - \frac{H(0,0)}{L_H^2}\right] \tag{72}$$

So if $L_m < L_H$, $H(0,t)$ can grow exponentially. We saw in section 5 and 6 that L_m is generally the diffusive scale. Since during linear evolution of $H(r,t)$, the current field correlations are generated from $M_L(r,t)$ through the alpha effect, we expect $L_H \sim L_m$. To estimate the maximum further growth of the current helicity, assume that we do have $L_m < L_H$, initially. Then for $t < t_1$ the current helicity grows exponentially with $H(0,t) = H(0,0)[(1-f_{mH})\exp(20aM_L(0)t/L^2) + f_{mH}]$, where $f_{mH} = L_m/L_H < 1$. At time t_1 we then have $H(0,t_1) = H(0,0)[(1-f_{mH})e^{5/2} + f_{mH}]$, where we have used the fact $\eta_{ambi} = \eta + 2aM_L(0) \sim 2aM_L(0)$. At later times, one can treat $H(0,t')$ in the integrand of Eq. (71) as a slowly varying function compared to $1/(1+4\eta_{ambi}(t-t')/L_m^2)^{7/2}$, pull it out of the integral and integrate the resulting equation to get

$$H(0,t) = H(0,t_1)\left(2 - \frac{1}{(1+4\eta_{ambi}(t-t_1)/L_m^2)^{5/2}}\right) \tag{73}$$

At large times one sees that $H(0,t) \to 2H(0,t_1) < 24H(0,0)$. The upper limit obtains only when $f_{mH} << 1$. A further growth of $H(0,t)$ in the nonlinear stage implies a further reduction of α_{eff}. Incase of purely ambipolar drift this reduction is still very small (cf. Eq. (65), while α_{GD} reduction depends on how large $f(l,t)$ grows before attaining the saturated state. For example if as argued in Paper I, $f(l,t) << 1$ in the saturated state then α_{GD} will also suffer only a modest reduction, due to small-scale dynamo action.

Before ending this section, we mention two other interesting non linear effects, which obtains under the action of ambipolar drift, which both invlove the existence of a non-zero magnetic helicity:

- (a) Relaxation through selective decay

  Suppose we start off with a random field configuration which has an initial non-zero magnetic helicity $P_H(0,t_i)$, and ask how it will evolve under the action of ambipolar drift. Due to ambipolar drift and the resulting ion-neutral friction, the field energy is constantly drained into heat. This is reflected by the diffusion terms in the equations for M_L and H. However, for a large conductivity, the magnetic helicity is almost conserved, reflecting the fact that the field is almost frozen into the ions. A non-zero magnetic helicity also implies neccessarily a minimum non-zero field energy. So the random field cannot evolve into a zero energy configuration, but rather into a minimum energy configuration conserving magnetic helicity. Such a selective decay of energy, conserving helicity is thought to lead to self organisation into larger scale structures (Taylor 1974). Our formalism incorporating ambipolar drift offers

a simple route to study the dynamics of this relaxation process, and the final relaxed state. A study of such relaxation trough selective decay, using the equations for M_L and H derived here is in progress and will be reported elsewhere.

- (b) Inverse cascade due to the non linear dynamo

  Suppose we had no kinetic helicity. But we had created large, random, small-scale fields, and also some small-scale magnetic (and current) helicity. For example, large random magnetic fields, correlated on small scales may be generated during phase transitions in the early universe. Further, in some of these phase transitions, like the electroweak one, there are speculations that large magnetic (and current) helicities may arise (Cornwall 1997, Joyce and Shaposhnikov 1997). Then our model non linear equation for the mean field (16), (and the work of Pouquet *et al.* 1976), indicates that the small-scale current helicity can lead to large-scale dynamo action. The alpha effect will be purely magnetic. This will lead to a coupling of small scale to large scale and "inverse cascade" (or dynamo growth) of magnetic energy to larger scales, due to purely nonlinear effects of the Lorentz force. If there were no other source of magnetic energy, the energy and current helicity would decay montonically. However the approximate conservation of magnetic helicty under near ideal MHD conditions, would still keep this dynamo active. It would be interesting to explore this issue further, using the full MHD equations,

9. Discussion and conclusions

We have revisited here the dynamics of fluctuating magnetic fields in turbulent fluids, ab initio. In doing this we have also incorporated the effects of ambipolar drift, as would obtain in a significantly neutral gas. Ambipolar drift introduces a magnetic field dependent addition to the velocity field in the induction equation. The resulting non linear equation may also be viewed, albiet with some caution, as a toy model for the MHD problem. Assuming that the velocity field has a turbulent component, we have derived the evolution equations for the mean and fluctuating magnetic field. These equations are used to discuss a number of astrophysically interesting problems. We summarise below the principle results of our work.

First, in the presence of ambipolar drift, the dynamo equation for the mean field, and the equations for the magnetic correlations are both modified. Assuming a Gaussian closure, one gets an extra diffusion term proportional to the energy density in the fluctuating fields and a reduction to the alpha effect due to the average current helicity of the fluctuating fields. These equations (Eq. (16) , Eq. (24) and Eq. (25)) form a closed set of nonlinear partial differential equations for the evolution of both the mean magnetic field and the magnetic fluctuations, incorporating the back reaction effects of ambipolar drift. Due to the Gaussian closure approximation, the nonlinearity appears as time-dependent co-efficients involving only the average properties of the fluctuating field itself.

We applied our equations in section 5 and 6 to discuss small-scale dynamo action in galaxies, assuming a Kolmogorov type galactic turbulence. In the kinematic phase, dynamo action exponentiates small-scale fields provided the MRN associated with the turbulence, exceeds a critical value $R_c \approx 60$. Further, for Kolmogorov type turblulence, the critical MRN for excitation of a mode extending upto $r \sim l$ is also $R_m(l) = R_c$, as expected from the scale invariance in the inertial range. In galaxies, it is likely that the ratio of the magnetic to fluid reynolds number $R_m/R_e >> 1$. In this case the fastest growing modes grow on a timescale comparable to the turn-over time of the smallest eddies at the cut-off scale l_c, with $\Gamma \sim V R_e^{1/2}/L \sim v_c/l_c$. Modes whose longitudinal correlation function extends upto $r \sim l > l_c$ grow at a slower rate $\Gamma \sim v_l/l$. Further, the field is strongly peaked about a region $r = r_d < l_c(R_m/R_e)^{-1/2}$ about the origin for all the modes. For the most rapidly growing mode, $w(r)$ changes sign accross $r = l_c$ and rapidly decays with increasing r/l_c. Interms of the Zeldovich rope-dynamo, one may picture, the small-scale field as being concentrated in ropy structures with thickness $\sim r_d$ and curved on a scale upto $\sim l$ for a mode extending to $r \sim l$.

As the small-scale fields grow, ambipolar drift adds to the diffusion coefficient and the effective MRN, R_{ambi} for fluid motion on any scale of the turbulence decreases. If $R_{ambi}(L)$ decreases to a value $R_c \sim 60$, dynamo action will stop completely. In a *sufficiently weakly ionised medium* ambipolar drift by itself can then lead to a saturation of small-scale fields before the turbulence is drastically modified. The required small ionisation fraction may obtain in the universe just after recombination and in proto stellar disks, but not in galactic gas. The effective magnetic Reynolds number in galactic gas, even including ambipolar diffusion, is much larger than R_c. In this case, as the the small-scale field grows in strength, it continues to be concentrated into thin ropy structures, as in the kinematic regime. In a companion paper (Subramanian, 1997 : Paper I) we have built upon the results obtained in sections 5 and 6, to discuss in detail how the small-scale dynamo may saturate in the galactic context, while preserving large-scale dynamo action. The crucial property of the small-scale dynamo generated field which allows this is its spatial intermittency. The field can build up locally to a level which will lead to small-scale dynamo saturation, while at the same time having a sub-equipartition *average* energy density, so that the diffusive property of the turbulence is not drastically affected.

We also considered in sections 7 and 8 the evolution of the current-field correlations. The dynamics of helical correlations has not been as extensively studied in the literature as that of longitudinal correlations (and the small-scale dynamo). In the kinematic phase the average current helicity associated with the small-scale field, H, decays due to diffusion, unless forced to grow by coupling to the exponentially growing longitudinal magnetic correlations. We showed in section 7, using a WKBJ approximation to the relevant Greens function, that the coupling with M_L, leads to an exponential growth of the current helicity $H(0,t)$ in the kinematic regime. So the assumption of small-scale stationarity made for example by Gruzinov and Diamond (1994) to derive a constraint on H is not valid. The growth of small-scale fields and hence $H(0,t)$ goes to decrease the effective alpha effect. The extent of the decrease depends on how much the small-scale field grows before it

saturates. In general we find the reduction in the alpha effect due to the ambipolar drift itself, as given by Eq. (65) to be negligible in galaxies. The alpha effect reduction due to the growing small scale field, in the quasi linear approximation to the full MHD employed by Gruzinov and Diamond (1994) is also small (Eq. (66) for α_{GD}), as long as $f(l,t) < 1$.

In section 8 we considered the non linear growth of the current helicity. We find that in principle $H(0,t)$ can grow further, due to its nonlinear coupling with M_L. This could lead to further reduction of α_{GD}. Even then, provided the small-scale dynamo saturates at sub-equipartition levels as argued in Paper I, the alpha effect is likely to be preserved until the large scale field itself grows to near equipartition level. Ofcourse, it is unclear what is the domain of validity of the quasilinear approach, and so this result must be treated with caution, until rederived in a full MHD treatment.

The equations for the magnetic correlations incorporating ambipolar drift derived here, may offer a simple route to study the dynamics of relaxation through selective decay. In the case when magnetic helicity (without kinetic helicity) is initially present, non linear effects due to the Lorentz force can also lead to a magnetic alpha effect and dynamo generation of large scale fields, as envisaged in the early work of Pouquet *et al.* (1976). It will be worthwhile to study these issues further, possibly with a numerical treatment of the equations derived here. It is also desirable to return to the dynamics of small scale fields in the full MHD context; Perhaps by finding simple ways of incorporating the dynamics of the velocity correlations on an equal footing as that of the magnetic correlations. We hope to return to these issues in the future.

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A. Evolution of fluctuating field correlations

We give below more algebraic details involved in simplifying the first term in Eq. (21) . This term is given by

$$< \int \ {}^{y}R_{jpq}\left(v_T^p(\mathbf{y},t)\ {}^{x}R_{ilm}(v_T^l(\mathbf{x},s)M_{mq})\right) ds >= -\epsilon_{itu}\epsilon_{ulm}\epsilon_{jrs}\epsilon_{spq}\frac{\partial^2}{\partial r^r \partial r^t}\left[T^{lp}M_{mq}\right] \qquad \text{(A1)}$$

For examining the evolution of M_L one needs to multiply the above equation by $r^i r^j/r^2$. We

can simplify the resulting equation by using the identity

$$r^i r^j \frac{\partial^2 A}{\partial r^r \partial r^t} = \frac{\partial^2 (A r^i r^j)}{\partial r^r \partial r^t} - \delta_{jt} r^i \frac{\partial A}{\partial r^r} - \delta_{ir} r^j \frac{\partial A}{\partial r^t} - \delta_{jt}\delta_{ir} A \tag{A2}$$

where $A = T^{lp} M_{mq}$. Then using $\epsilon_{itu}\epsilon_{ulm} = \delta_{il}\delta_{tm} - \delta_{im}\delta_{tl}$, and the definition of T_{LL}, T_{NN} and C, straightforward algebra gives the contribution of the first term to $(\partial M_L/\partial t)$

$$\frac{\partial M_L}{\partial t}(1\text{ st term}) = -\frac{1}{r^4}\frac{\partial}{\partial r}(r^4 T_{LL}\frac{\partial M_L}{\partial r}) + \frac{G}{2} M_L + 4CH \tag{A3}$$

The second term of Eq. (21) gives an identical contribution.

To derive the evolution of H due to these terms multiply Eq. (A1) by $\epsilon_{ijf} r^f$. Using the fact that the turbulent velocity and small scale field have vanishing divergence, we have $M_{ij,j} = 0$ and $T_{ij,j} = 0$. This allows one to simplify the contribution from the first term to

$$\frac{\partial H}{\partial t}(1\text{ st term}) = -\frac{\epsilon_{ijf} r^f}{2r^2}\left[T_{ij,tr} M_{tr} + T_{tr} M_{ij,tr} - T_{ir,t} M_{tj,r} - T_{tj,r} M_{ir,t}\right] \tag{A4}$$

The first two terms on the RHS of Eq. (A4) can be further simplified by noting that $\epsilon_{ijf} T_{ij} = 2Cr^f$ and $\epsilon_{ijf} M_{ij} = 2Hr^f$. We have then

$$-\frac{\epsilon_{ijf} r^f}{2r^2}\left[T_{ij,tr} M_{tr} + T_{tr} M_{ij,tr}\right] = -[T_{LL} H'' + T'_{LL} H' + \frac{4T_{LL} H'}{r} + M_L C'' + M'_L C' + \frac{4M_L C'}{r}] \tag{A5}$$

Here prime denotes a derivative with respect to r. To evaluate the contribution of the last two terms on the RHS of Eq. (A4) it is convenient to split up the tensors M_{ij} and T_{ij} into symmetric and antisymmetric parts (under the interchange of (ij)). We put a superscript S on the symmetric part and A on the antisymmetric part. Then we can write after some algebra

$$\begin{aligned} \frac{\epsilon_{ijf} r^f}{2r^2}\left[T_{ir,t} M_{tj,r} + T_{tj,r} M_{ir,t}\right] &= \frac{\epsilon_{ijf} r^f}{r^2}\left[T^S_{ir,t} M^A_{tj,r} + T^A_{ir,t} M^S_{tj,r}\right] \\ &= -\left[H T''_{LL} + C M''_L + T'_{LL} H' + M'_L C' + \frac{4H T'_{LL}}{r} + \frac{4C M'_L}{r}\right] \end{aligned} \tag{A6}$$

Adding the contributions from Eq. (A5) and (A6) gives

$$\frac{\partial H}{\partial t}(1\text{ st term}) = -\frac{1}{r^4}\frac{\partial}{\partial r}(r^4 \frac{\partial}{\partial r}[T_{LL} H + C M_L] \tag{A7}$$

The second term of Eq. (21) gives an identical contribution.

B. The WKBJ analysis of the kinematic small-scale dynamo

First, in order to implement the boundary condition at $r = 0$, under WKBJ approximation, it is better to transform to a new radial co-ordinate x, where $r = e^x$. Also to eliminate first derivative terms in the resulting equation we substitute $\Phi(x) = \exp(x/2)\Theta$ and get

$$\frac{d^2\Theta}{dx^2} + p(x)\Theta = 0 \tag{B1}$$

where

$$p(x) = \frac{-(\Gamma + U)e^{2x}}{\kappa_N} - \frac{1}{4} \tag{B2}$$

The WKBJ solutions to this equation are linear combinations of

$$\Theta = \frac{1}{p^{1/4}} \exp(\pm i \int^x p^{1/2} dx) \tag{B3}$$

The solutions have to satisfy the boundary conditions $\Theta(x) \to 0$ for $x \to \pm\infty$. One therefore has a standard WKBJ eigenvalue problem for the determination of Γ. Note that as $x \to -\infty$, $p \to -9/4$ and so the WKBJ solutions are in the form of growing and decaying exponentials at this end. This also the case as $x \to +\infty$ since $p \to -\Gamma e^{2x} < 0$ for growing mode solutions with $\Gamma > 0$. In order to match the boundary conditions at both ends, the solution has to be the growing exponential at $x = -\infty$ and transit to the decaying exponential as $x \to +\infty$. This can only obtain if the $p(x)$ goes through zeros, by U becoming negative for some range of r. For U considered here, in general, $p(x)$ goes through two zeros. Suppose these occur at x_1 and x_2 with $x_1 < x_2$. Then the WKBJ soultions will be oscillatory in the range $x_1 < x < x_2$. The requirement that the oscillatory solutions match on to the growing exponential near $x = -\infty$ and the decaying exponential as $x \to +\infty$, gives the standard condition (cf. Jeffreys and Jeffreys 1966, Mestel & Subramanian 1991) on the the eigenvalue Γ

$$\int_{x_1}^{x_2} p^{1/2}(x) dx = \frac{(2n+1)\pi}{2}. \tag{B4}$$

One also determines the eigenfunction $\Theta(x)$, under the WKBJ approximation, to be

$$\begin{aligned} \Theta &= \frac{A}{(-p)^{1/4}} \exp\left[\int_{x_1}^{x} (-p)^{1/2} dx\right] \qquad && x < x_1 \\ &= \frac{2^{1/2} A}{p^{1/4}} \sin\left[\int_{x_1}^{x} (p)^{1/2} dx + \frac{\pi}{4}\right] \qquad && x_1 < x < x_2 \\ &= \frac{(-1)^n A}{(-p)^{1/4}} \exp\left[-\int_{x_2}^{x} (-p)^{1/2} dx\right] \qquad && x_2 < x \end{aligned} \tag{B5}$$

B.1. Critical MRN for the marginal mode

Let us first use Eq. (B4) to determine the critical value of $R_m = R_c$ needed for growth of the small-scale fields. For this we have to put $\Gamma = 0$ and find R_m which satisfies (B4). With $\Gamma = 0$, using U from (39) , defining $y = r/L$ we have in the range $l_c/L < y < 1$,

$$p(y) = \frac{-(9/4) - (29R_m/54)y^{4/3} + (13R_m^2/108)y^{8/3}}{(1 + y^{4/3}R_m/3)^2} \tag{B6}$$

We will find that $R_m = R_c$ is large enough that one can assume $1/R_m << y^{4/3}$ in the above expression for p. In this case the zeros occur approximately at $r_1/L \sim (174/39R_c)^{3/4} = y_0$ and at

$r_2/L \sim 1$. (Here $r_1 = \exp(x_1)$ and $r_2 = \exp(x_2)$). The integral condition then becomes

$$\int_{x_1}^{x_2} p^{1/2}(x)dx = \int_{y_0}^{1} [\frac{13}{12} - \frac{174}{39R_c y^{4/3}}]^{1/2}\frac{dy}{y} = \frac{\pi}{2} \tag{B7}$$

where we have looked for the critical reynolds number for the principal mode with $n = 0$. The integral in Eq. (B7) can be done exactly and gives the condition

$$\frac{3}{2}(\frac{13}{12})^{1/2}\left[ln[\frac{1+\sqrt{1-y_0^2}}{y_0}] - \sqrt{1-y_0^2}\right] = \frac{\pi}{2} \tag{B8}$$

The solution of this equation implies a critical value for the excitation of the marginal mode $R_m = R_c \approx 60$. Note that R_c was estimated by taking $n = 0$ in Eq. (B4). One can easily get also the limiting magnetic reynolds number needed for the exitation of higher order modes. It should also be pointed out that in Eq. (B6) for $p(y)$, $y^{4/3}R_m = (r/L)^{4/3}R_m \equiv (r/l)^{4/3}R_m(l)$. So the above equations determening R_c are the same if we replace (L, R_m) by $(l, R_m(l))$. This shows that the critical MRN for excitation of a mode extendinng to $r \sim l$ is also $R_m(l) = R_c$, as expected from the scale invariance in the inertial range.

The actual value of the MRN at the outer scale is likely to be much larger than R_c, in galaxies. Infact it is likely that $R_m/R_e >> 1$. Let us now estimate the growth rate of the fastest growing mode in this case.

B.2. Growth rate for the fastest growing mode

For this we fix R_m and R_e and look for the value of Γ which satisfies Eq. (B4) . For $R_m >> R_e$,the potential U is negative at $r = L$, decreases monotonically from $r = L$ to $r = l_c$ and is still negative at $r = l_c$. It only starts increasing for $r < l_c$. The fastest growing mode is then expected to concentrate at $r < l_c$. Also the turning point corresponding to $x = x_1$ occurs at $r < l_c$ for all the modes. So to determine the growth rate of the fastest growing mode and also examine the structure of the modes at small radius, one must adopt the form of $T_{LL}(r)$ with $r < l_c$, given in Eq. (38). For $r < l_c$, one then has $\kappa_N = \eta[1 + (1/3)(r/r_d)^2]$ and

$$U(r) = \frac{V}{3L}R_e^{1/2}\frac{[2z^{-2} - 1 - 4z^2]}{[1 + z^2]} \tag{B9}$$

Here $z = r/(\sqrt{3}r_d)$ with

$$r_d = \frac{l_c}{R_m^{1/2}(l_c)} \tag{B10}$$

setting the scale over which the potential varies. Using this form of the potential we then have

$$p(z) = \frac{A_0 z^4 - B_0 z^2 - 9/4}{(1 + z^2)^2} \tag{B11}$$

for the range $0 < r < l_c$. Here $\bar{\Gamma} = \Gamma/(VR_e^{1/2}/3L)$ is a normalised growth rate, $A_0 = 15/4 - \bar{\Gamma}$ and $B_0 = \bar{\Gamma} - 1/2$.

For the above form of $p(z)$, there is only one real positive zero z_0. It turns out that z_0 is large enough such that the 9/4 in $p(z)$ above can be neglected compared to the other terms giving $z_0 \approx (\bar{\Gamma} - 1/2)^{1/2}/(15/4 - \bar{\Gamma})^{1/2}$. Note that for the form of $T_{LL}(r)$ we have adopted the potential is discontinuous at $r = l_c$. For $r = l_c - \epsilon$, $U = U^- = -4(VR_e^{1/2}/3L)$ while for $r = l_c + \epsilon$, $U = U^+ = -4/3(VR_e^{1/2}/3L)$. As we mentioned earlier this does not alter the results qualitatively since they are generally dependent on integrals over U. It means however that for $4/3 < \bar{\Gamma} < 4$, which we will see obtains for the fastest growing mode, the outer zero of p is at $r = l_c$ or $z = z_c = l_c/(\sqrt{3}r_d) = (R_m/3R_e)^{1/2}$. For $\bar{\Gamma}$ corresponding to the fastest growing mode, we then obtain the integral condition

$$\int_{z_0}^{z_c} \frac{[A_0 z^4 - B_0 z^2]^{1/2}}{z^2} \frac{dz}{z} = \frac{\pi}{2} \tag{B12}$$

In the above we have again assumed that $z_0^2 >> 1$, which we will show to be true below. This integral can be done exactly and gives the condition

$$A_0^{1/2} \ln[2A_0 z_c^2/B_0 - 1] - 2(A_0 z_c^2 - B_0)/z_c = \frac{\pi}{2}. \tag{B13}$$

In the above condition, since $z_c^2 = (R_m/3R_e) >> 1$, one can get a good iterative approximate solution for $\bar{\Gamma}$. In the first approximation one neglects the constant terms compared to the z_c^2 terms to get

$$\bar{\Gamma} \approx 15/4 - (\pi/2ln(R_m/R_e))^2. \tag{B14}$$

For example for $R_m/R_e = 10^{15}$ one has $\bar{\Gamma} \sim 3.748$ or $\Gamma \sim 1.25VR_e^{1/2}/L$. So as advertised the fastest growing modes grow on a timescale comparable to the turn-over time of the smalles, cut-off scale eddies, if $R_m >> R_e$, with $\Gamma \sim VR_e^{1/2}/L$. One can also go back and check, using this value of Γ that $z_0 \sim 30 >> 1$ and so the approximations made assuming $z_0, z_c >> 1$ are very good.

B.3. Spatial structure of the eigen modes

We now briefly consider the spatial structure for various eigenmodes of the small-scale dynamo, for the case $R_m(l_c) = R_m/R_e >> 1$. In this case as we mentioned earlier, the turning point corresponding to $x = x_1$ occurs at $r < l_c$ for all the modes. The eigenfunction for $r < l_c$ are then given by Eq. (B5) with p given by the form in (B11) .

Infact near the origin, one can use the the original equation (37) to find the behavior of M_L and $w(r)$. One finds that for small r, Φ satisfies the equation

$$\frac{d^2\Phi}{dz^2} - \frac{2\Phi}{z^2} + \alpha\Phi = 0 \tag{B15}$$

where the constant $\alpha = (5 - \bar{\Gamma})$. The solution can be found by elementary methods $\Phi = z^2 \sin(\sqrt{\alpha} z)/(\sqrt{\alpha} z)$. So then have

$$M_L(r,t) = \frac{M_L(0,t)}{[1+z^2]^{1/2}} \frac{\sin(\sqrt{\alpha}z)}{\sqrt{\alpha}z} \tag{B16}$$

A similar result can also be found from the WKBJ solution, where one gets for $z < z_0$,

$$M_L = \frac{e^{\Gamma t} F(z)}{z^{3/2}(1+z^2)^{1/2}} \tag{B17}$$

with

$$F(z) = \frac{A_1}{(-p(z))^{1/4}} \exp\left[\int_{z_0}^{z} (-p(z))^{1/2} \frac{dz}{z}\right] \tag{B18}$$

As $z \to 0$, $-p(z) \to 9/4 - \alpha z^2$ and so this soln goes over to

$$M_L = \frac{M_L(0,t)}{[1+z^2]^{1/2}} \frac{1}{(1-(4\alpha/9)z^2)} \exp[\frac{-\alpha z^2}{6}] \tag{B19}$$

This is in good agreement with the exact soln determined for small z. For larger z, M_L decreases monotonically with z. The WKBJ treatment gives for $z_0 < z < z_2$ (z_2 is the outer turning point)

$$F(z) = \frac{2^{1/2} A_1}{(p(z))^{1/4}} \sin\left[\int_{z_0}^{z} (p(z))^{1/2} \frac{dz}{z} + \frac{\pi}{4}\right] \tag{B20}$$

Away from $z = z_0$, the z^4 term in $p(z)$ dominates and $p \to (15/4 - \bar{\Gamma}) = A_0$. So for $z >> z_0$ we have

$$M_L \approx \frac{e^{\Gamma t} 2^{1/2} A_1}{A_0^{1/4} z^{5/2}} \sin\left[A_0^{1/2} ln(\frac{z}{z_0}) + \frac{\pi}{4}\right] \tag{B21}$$

The above equations show that M_L decreases rapidly with increasing z. Since $z = r/r_d$, one sees that M_L and hence $w(r)$ for all the modes are strongly peaked about the radius $r \sim r_d = l_c/R_m^{1/2}(l_c) << l_c$, for the case $R_m(l_c) >> 1$.

We also mentioned that $w(r)$ must become negative at some radius. From the WKBJ solution it is apparent that the number of zero crossings for the WKB solution will depend on the order n of the mode. Let us consider the fastest growing mode. We saw earlier that for this mode the outer turning point is at $r = l_c$ or $z = z_c$. For this mode one can check that from the WKBJ solution that $w(r) > 0$, in the region $0 < r < l_c$. For $r > l_c$ and $R_m/R_e >> 1$, one has from Eq. (39) , $U(r) \approx -(4VR_e^{1/2}/9L)(r/l_c)^{-2/3}$ and $\kappa_N \approx (VL/3R_e)(r/l_c)^{4/3}$. Therefore

$$p(r) \approx \frac{13}{12} - (\frac{\Gamma}{(v_l/3l)})(\frac{r}{l})^{2/3} \quad \text{for } l_c < r < L;\ R_m/R_e >> 1 \tag{B22}$$

For the fastest growing mode one can igmore the constant term and get $-p(r) \approx \bar{\Gamma}(r/l_c)^{2/3}$. Then

$$M_L(r) \propto (\frac{l_c}{r})^{7/3} \exp\left[-3\bar{\Gamma}^{1/2}((r/l_c)^{1/3} - 1)\right] \tag{B23}$$

and

$$w \propto \frac{1}{y^2}\left(\frac{2}{3}(\frac{l_c}{r})^{1/3} - \bar{\Gamma}^{1/2}\right) \exp\left[-3\bar{\Gamma}^{1/2}((r/l_c)^{1/3} - 1)\right] \tag{B24}$$

For $\bar{\Gamma} \sim 3.75$ we can see that $w(r) < 0$ for $r > l_c$ and its modulus decreases to zero rapidly with increasing r/l_c. So w changes sign accross the transition point $r = l_c$ for the form of longitudinal correlation function T_{LL} we have adopted. For modes with a smaller growth rate Γ, we see from Eq. (B22), that the outer turning point, got by putting $p(r) = 0$, is at $r/l \sim (13/(12\bar{\Gamma}_l))^{3/2}$, where $\bar{\Gamma}_l = \Gamma/(v_l/3l)$. So modes with growth rate $\Gamma \sim v_l/l$, extend upto $r \sim l$.

In summary, in the case $R_m/R_e >> 1$, $w(r)$ is strongly peaked about a region $r < l_c(R_m/R_e)^{-1/2}$ about the origin for all the modes. For the most rapidly growing mode, $w(r)$ changes sign accross $r = l_c$ and rapidly decays with increasing r/l_c. (If one had adopted a sufficiently smooth form for $T_{LL}(r)$ around $r = l_c$, the value where w changes sign would still be $r \sim l_c$, but could have been better determined by the WKBJ analysis). Also slower growing modes with $\Gamma \sim v_l/l$, extend upto $r \sim l$. A more thorough analysis of the eigenfunctions can be found in Kleeorin et. al. (1986), for the simple case when the longitudinal velocity correlation function has only a single scale. Their analysis is also applicable to the mode near the cut-off scale of Kolmogorov type turbulence.

Let us now consider the corresponding eigenfunction for the marginal mode. In this case, as we saw earlier, the inner turning point occurs at $r = r_1 = Ly_0 \sim 0.14L$ and the outer turning point is at $r = r_2 \sim L$. The WKBJ solution can be used to determine the the eigenfunction. We get for $r < r_1$,

$$\Theta(y) = \frac{A_2}{(-p(y))^{1/4}} \exp(\int_0^y (-p(y'))^{1/2} \frac{dy'}{y'}) \tag{B25}$$

where $p(y)$ is given by Eq. (B6) . For $r << L$, we can neglect the $y^{8/3}$ term in have Eq. (B6) compared to the constant and $y^{4/3}$ terms. This implies $\Theta(y) \approx y^{3/2} \exp(-13R_m y^{4/3}/54)$ and so for the marginal mode

$$M_L(r,t) \propto \frac{\exp(-(13/54)R_m y^{4/3})}{(1 + (1/3)R_m y^{4/3})^{1/2}} \tag{B26}$$

One sees that the eigenfunction is concentrated in a radius $r \sim L/R_c^{3/4} = r_c$ for the marginal mode. Infact if we define $\bar{z}^2 = R_c y^{4/3}/3$, then for the marginal mode with $\Gamma = 0$ we have from Eq. (B6), $p = (-9/4 - 25\bar{z}^2/18 + 13\bar{z}^4/12)/(1 + \bar{z}^2)^2$. This is very similar to the $p(z)$ defined in Eq. (B11) except for the identifications $z \to \bar{z}$, $A_0 \to 13/12$, $B_0 \to 25/18$. Going through the same analysis as for the $p(z)$ of Eq. (B11) , gives the properties of of $w(r)$ for the marginal mode : For the marginal mode $w(r)$ peaks within $\bar{z} \sim 1$, corresponding to a radius $r \sim L/R_c^{3/4}$, changes sign to become negative at $r \sim L$ and dies rapidly for larger r/L.

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